Spherical gradient manifolds

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by

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Abstract. — We study the action of a real-reductive group $G = K\exp(\mathfrak{p})$ on real-analytic submanifold X of a Kähler manifold Z. We suppose that the action of G extends holomorphically to an action of the complexified group $G^{\mathbb{C}}$ such that the action of a maximal Hamiltonian subgroup is Hamiltonian. The moment map μ induces a gradient map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$. We show that $\mu_{\mathfrak{p}}$ almost separates the K–orbits if and only if a minimal parabolic subgroup of G has an open orbit. This generalizes Brion's characterization of spherical Kähler manifolds with moment maps.

Résumé. — Nous étudions l'action d'un groupe réel-réductif $G = K\exp(\mathfrak{p})$ sur une sous-variété réel-analytique X d'une variété kählérienne Z. Nous supposons que l'action de G peut être prolongée à une action holomorphe du groupe complexifié $G^{\mathbb{C}}$ telle que l'action d'un sous-groupe maximal compact de $G^{\mathbb{C}}$ soit hamiltonienne. L'application moment μ induit une application gradient $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$. Nous montrons que $\mu_{\mathfrak{p}}$ separe les orbites de K si et seulement si un sous-groupe minimal parabolique de G possède une orbite ouverte dans X. Ce résultat généralise la charactérisation de Brion des variétés kählériennes sphériques qui admettent une application moment.

1. Introduction

Let $U^{\mathbb{C}}$ be a complex-reductive Lie group with compact real form U and let Z be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically such that U acts by Kähler isometries. Assume furthermore that the U–action on Z is Hamiltonian, i. e. that there exists a U–equivariant moment map $\mu\colon Z \to \mathfrak{u}^*$ where $\mathfrak{u}$ denotes the Lie algebra of U.

In the special case that Z is compact it is shown in [**Bri87**] (see also [**HW90**]) that μ separates the U–orbits if and only if Z is a spherical $U^{\mathbb{C}}$–manifold, which means that a Borel subgroup of $U^{\mathbb{C}}$ has an open orbit in Z. Note that μ separates the U–orbits if and only if it induces an injective map $Z/U \hookrightarrow \mathfrak{u}/U$. Moreover, this is equivalent to the property that the U–action on Z is coisotropic.

In this paper we generalize Brion's result to actions of real-reductive groups on real-analytic manifolds which moreover are not assumed to be compact. More precisely, we consider a closed subgroup G of $U^{\mathbb{C}}$ which is compatible with the Cartan decomposition $U^{\mathbb{C}} = U\exp(i\mathfrak{u})$. This means that $G = K\exp(\mathfrak{p})$ where $K := G \cap U$ and $\mathfrak{p}$ is an $\mathrm{Ad}(K)$–invariant subspace of

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$i\mathfrak{u}$. Let X be a G–invariant real-analytic submanifold of Z. By restriction, the moment map μ induces a K–equivariant gradient map $\mu_{\mathfrak{p}}\colon X \to (i\mathfrak{p})^*$.

There are two main differences between the complex and the real situation: Even if X is connected an open G–orbit in X does not have to be dense and in general the fibers of $\mu_{\mathfrak{p}}$ are not connected. Therefore one cannot expect $\mu_{\mathfrak{p}}$ to separate the K–orbits globally in X. We say that $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits if there exists a K–invariant open subset Ω of X such that $K \cdot x$ is open in $\mu_{\mathfrak{p}}^{-1}\big(K \cdot \mu_{\mathfrak{p}}(x)\big)$ for all $x \in \Omega$. Geometrically this means that the induced map $\Omega/K \to \mathfrak{p}/K$ has discrete fibers. If $\Omega = X$, we say that $\mu_{\mathfrak{p}}$ almost separates the K–orbits in X.

We suppose throughout this article that X/G is connected. Now we can state our main result.

Theorem 1. — *The following are equivalent.*

1. *The gradient map $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits.*
2. *The gradient map $\mu_{\mathfrak{p}}$ almost separates the K–orbits in X.*
3. *The minimal parabolic subgroup Q_0 of G has an open orbit in X.*

Hence, Theorem 1 gives a sufficient condition on the G–action for $\mu_{\mathfrak{p}}$ to induce a map $X/K \to \mathfrak{p}/K$ whose fibers are discrete, while on the other hand the gradient map yields a criterion for X to be spherical. Moreover we see that sphericity is independent of the particular choice of $\mu_{\mathfrak{p}}$, i. e. if one gradient map for the G–action on X generically separates the K–orbits in X, then this is true for every gradient map.

Let us outline the main ideas of the proof. First we observe that X contains an open Q_0–orbit if and only if $(G/Q_0) \times X$ contains an open G–orbit with respect to the diagonal action of G. The gradient map $\mu_{\mathfrak{p}}$ on X induces a gradient map $\widetilde{\mu}_{\mathfrak{p}}$ on $(G/Q_0) \times X$. Now we are in a situation where we can apply the methods introduced in [**HS07b**]. These allow us to show that open G–orbits correspond to isolated minimal K–orbits of the norm squared of $\widetilde{\mu}_{\mathfrak{p}}$. In order to relate the property that $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits to the existence of an isolated minimal K–orbit, we need the following result. We consider the restriction $\mu_{\mathfrak{p}}|_{K\cdot x}\colon K \cdot x \to K \cdot \mu_{\mathfrak{p}}(x)$ which is a smooth fiber bundle with fiber $K_{\mu_{\mathfrak{p}}(x)}/K_x$. In the special case $G = K^{\mathbb{C}}$ it is proven in [**GS84**] that for generic x the fiber $K_{\mu_{\mathfrak{p}}(x)}/K_x$ is a torus. As a generalization we prove the following proposition, which also allows us to extend the notion of “K–spherical” defined in [**HW90**] to actions of real-reductive groups.

Proposition 2. — *Let $x \in X$ be generic and choose a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing $\mu_{\mathfrak{p}}(x)$. Then the orbits of the centralizer $\mathcal{Z}_K(\mathfrak{a})$ of $\mathfrak{a}$ in K are open in $K_{\mu_{\mathfrak{p}}(x)}/K_x$.*

These arguments yield the existence of an open Q_0–orbit under the assumption that $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits. For the other direction we apply the shifting technique for gradient maps.

Notice that our proof of Brion’s theorem is different from the ones in [**Bri87**] and [**HW90**]. In particular, for every generic element $x \in X$ we construct a minimal parabolic subgroup Q_0 of G such that $Q_0 \cdot x$ is open in X.

At present we do not know whether a spherical G–gradient manifold does only contain a finite number of G– and Q_0–orbits (which is true in the complex-algebraic situation). These and other natural open questions will be addressed in future works.

2. Gradient manifolds

In this section we review the necessary background on G–gradient manifolds and gradient maps. We then define what it means that a gradient map locally almost separates the orbits of a maximal compact subgroup of G and discuss several examples where this can be shown to be true.

2.1. The gradient map. — Here we recall the definition of the gradient map. For a detailed discussion we refer the reader to [**HS07b**].

Let U be a compact Lie group and $U^{\mathbb{C}}$ its universal complexification (see [**Ho65**]). We assume that Z is a Kähler manifold with a holomorphic action of $U^{\mathbb{C}}$ such that the Kähler form is invariant under the action of the compact real form U of $U^{\mathbb{C}}$. We assume furthermore that the action of U is Hamiltonian, i. e. that there exists a moment map $\mu\colon Z \to \mathfrak{u}^*$, where $\mathfrak{u}^*$ is the dual of the Lie algebra of U. We require μ to be real-analytic and U–equivariant, where the action of U on $\mathfrak{u}^*$ is the coadjoint action.

The complex reductive group $U^{\mathbb{C}}$ admits a Cartan involution $\theta\colon U^{\mathbb{C}} \to U^{\mathbb{C}}$ with fixed point set U. The -1-eigenspace of the induced Lie algebra involution equals $\mathrm{i}\mathfrak{u}$. We have an induced Cartan decomposition, i. e. the map $U \times \mathrm{i}\mathfrak{u} \to U^{\mathbb{C}}$, $(u,\xi) \mapsto u\exp(\xi)$ is a diffeomorphism. Let G be a θ-stable closed real subgroup of $U^{\mathbb{C}}$ with only finitely many connected components. Equivalently, we assume that G is a closed subgroup of $U^{\mathbb{C}}$, such that the Cartan decomposition restricts to a diffeomorphism $K \times \mathfrak{p} \to G$, where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{i}\mathfrak{u}$. In this paper such a group $G = K\exp(\mathfrak{p})$ is called *real-reductive*. Note that $U^{\mathbb{C}}$ itself is an example for such a subgroup G of $U^{\mathbb{C}}$.

Let X be a G–invariant real-analytic submanifold of Z such that X/G is connected. We identify $\mathfrak{u}$ with $\mathfrak{u}^*$ by a U–invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{u}$. Moreover we identify $\mathfrak{u}$ and $\mathrm{i}\mathfrak{u}$ by multiplication with i. Then the moment map $\mu\colon Z \to \mathfrak{u}^*$ restricts to a real-analytic map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ which is defined by $\big\langle \mu_{\mathfrak{p}}(x), \xi\big\rangle = \mu(x)(-\mathrm{i}\xi)$ for $\xi \in \mathfrak{p}$. We call $\mu_{\mathfrak{p}}$ a *G-gradient map on X* and we say that X is a *G-gradient manifold.* Note that $\mu_{\mathfrak{p}}$ is K–equivariant with respect to the adjoint action of K on $\mathfrak{p}$. In the special case $G = U^{\mathbb{C}}$, the gradient map coincides with the moment map up to the identification of $\mathfrak{u}^*$ with $\mathrm{i}\mathfrak{u}$.

In this paper, we consider real-analytic gradient maps which *locally almost separate the K–orbits.* By this, we mean that there exists a K–invariant open subset Ω of X such that the following equivalent conditions are satisfied.

1. $K \cdot x$ is open in $\mu_{\mathfrak{p}}^{-1}\big(K \cdot \mu_{\mathfrak{p}}(x)\big)$ for all $x \in \Omega$.
2. $K_{\mu_{\mathfrak{p}}(x)} \cdot x$ is open in $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)$ for all $x \in \Omega$.
3. The induced map $\overline{\mu}_{\mathfrak{p}}\colon \Omega/K \to \mathfrak{p}/K$ has discrete fibers.

If $\Omega = X$, we say that $\mu_{\mathfrak{p}}$ *almost separates the K–orbits.* We will show later that the set Ω on which $\mu_{\mathfrak{p}}$ almost separates the K–orbits can always be chosen to be X, i. e. $\mu_{\mathfrak{p}}$ separates locally almost the K–orbits if and only if $\mu_{\mathfrak{p}}$ almost separates them. If $\mu_{\mathfrak{p}}^{-1}\big(K \cdot \mu_{\mathfrak{p}}(x)\big) = K \cdot x$ for all $x \in X$, then we say that $\mu_{\mathfrak{p}}$ *globally separates the K–orbits.*

Lemma 2.1. — *Suppose that $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ locally almost separates the K–orbits. Then G has an open orbit in X.*

Proof. — By assumption there exists a K–invariant open subset $\Omega \subset X$ such that $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)^0 \subset K \cdot x$ holds for all $x \in \Omega$. Since $\mu_{\mathfrak{p}}$ is real-analytic, we find a point $x \in \Omega$ such that $\mu_{\mathfrak{p}}$ has maximal rank in x. We conclude from Lemma 5.1 in [**HS07b**] that $(\mathfrak{p}\cdot x)^{\perp} = T_x\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big) \subset \mathfrak{k} \cdot x$ and thus obtain

$$T_xX = (\mathfrak{p}\cdot x) \oplus (\mathfrak{p}\cdot x)^{\perp} \subset (\mathfrak{p}\cdot x) + (\mathfrak{k}\cdot x) = \mathfrak{g}\cdot x,$$

which means that $G \cdot x$ is open in X. $\square$

2.2. Examples. — In general, it is very difficult to verify directly that a G–gradient map separates (locally almost) the K–orbits. In this subsection we give some examples of situations where this can be done.

Example. — The connected group $G = K\exp(\mathfrak{p})$ acts on itself by left multiplication. The standard gradient map for this action is given by $\mu_{\mathfrak{p}}\colon G \to \mathfrak{p}$, $\mu_{\mathfrak{p}}\big(k\exp(\xi)\big) = \mathrm{Ad}(k)\xi$. Let $x_0 = k_0\exp(\xi_0) \in G$ be given. One checks directly that $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big) = x_0K$. Hence, $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits if and only if there exists a K–invariant open subset $\Omega \subset G$ such that $xK = Kx$ for all $x \in \Omega$. We claim that this is the case if and only if $\mathfrak{p}^K = \mathfrak{p}$.

Suppose that $xK = Kx$ holds for all x in a K–invariant open subset $\Omega \subset G$. This means that the fixed point set $(G/K)^K$ has non-empty interior. Since G/K is K–equivariantly diffeomorphic to $\mathfrak{p}$ with the adjoint K–action, we see that $\mathfrak{p}^K$ has non-empty interior and thus $\mathfrak{p}^K = \mathfrak{p}$.

Conversely, if $\mathfrak{p}^K = \mathfrak{p}$, then we have for every $x = k\exp(\xi) \in G$ that $Kx = K\exp(\xi) = \exp(\xi)K = xK$ holds.

Example. — We describe a class of totally real G–gradient manifolds where $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits.

Let (Z,ω) be a Kähler manifold endowed with a holomorphic $U^{\mathbb{C}}$–action such that the U–action is Hamiltonian with moment map $\mu\colon Z \to \mathfrak{u}^*$. Suppose that the action is defined over $\mathbb{R}$ in the following sense. There exists an antiholomorphic involutive automorphism $\sigma\colon U^{\mathbb{C}} \to U^{\mathbb{C}}$ with $\sigma\theta = \theta\sigma$ and there is an antiholomorphic involution $\tau\colon Z \to Z$ with $\tau^*\omega = -\omega$ and $\tau(g\cdot z) = \sigma(g)\cdot\tau(z)$ for all $g \in U^{\mathbb{C}}$ and all $z \in Z$. Consequently, the fixed point set $X := Z^\tau$ is a Lagrangian submanifold of Z and the compatible real form $G = K\exp(\mathfrak{p}) = (U^{\mathbb{C}})^\sigma$ acts on X. Let $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ be the K–equivariant gradient map induced by μ.

We claim that if μ locally almost separates the U–orbits in Z, then $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits in X. This claim is a consequence of the following three observations:

1. If μ locally almost separates the U–orbits, then μ separates all the U–orbits in Z (see [**HW90**]).
2. Since X is Lagrangian, we see that $\mu_{\mathfrak{k}}|_X \equiv 0$, where $\mu_{\mathfrak{k}}$ denotes the moment map for the K–action on Z. Note that under our identification we have $\mu = \mu_{\mathfrak{k}} + \mu_{\mathfrak{p}}$.
3. For every $x \in X$ the orbit $K\cdot x$ is open in $(U\cdot x)\cap X$.

Locally injective gradient maps separate locally almost the K–orbits. A class of G–gradient manifolds for which $\mu_{\mathfrak{p}}$ is locally injective is described in the following example.

Example. — Let $Z = U/K$ be a Hermitian symmetric space of the compact type, and let $G = K\exp(\mathfrak{p})$ be a Hermitian real form of $U^{\mathbb{C}}$. Then Z is a G–gradient manifold and every gradient map $\mu_{\mathfrak{p}}\colon Z \to \mathfrak{p}$ is locally injective. Consequently, $\mu_{\mathfrak{p}}$ separates locally almost the K–orbits in Z.

We will elaborate a little bit on further properties of $\mu_{\mathfrak{p}}\colon Z \to \mathfrak{p}$. Let $\tau\colon Z \to Z$ be the holomorphic symmetry which fixes the base point $z_0 = eK$. Then we have $Z^\tau = \mu_{\mathfrak{p}}^{-1}(0)$. Moreover, one can show that Z^τ is a K–invariant closed complex submanifold of Z and that every K–orbit in Z^τ is open in Z^τ. Furthermore, $K^{\mathbb{C}}$ acts on Z^τ and we have $K^{\mathbb{C}}\cdot z = K\cdot z$ if and only if $z \in Z^\tau$ holds. Finally, note that $\mu_{\mathfrak{k}}$ separates all K–orbits in Z.

3. Spherical gradient manifolds and coadjoint orbits

As we have remarked above it is very hard to verify directly if a given gradient map defined on X separates the K–orbits. The main result of this paper states that this is true if and only if X is a spherical gradient manifold. Hence, this is independent of the particular choice of a gradient map $\mu_{\mathfrak{p}}$.

In this section we give the definition of spherical gradient manifolds. For this we first review the definition of minimal parabolic subgroups. After that, we discuss the orbits of the adjoint K–action on $\mathfrak{p}$ which are the right analogues of complex flag varieties.

We continue the notation of the previous section: Let $G = K\exp(\mathfrak{p})$ be a closed compatible subgroup of $U^{\mathbb{C}}$ and let X be a real-analytic G–gradient manifold with K–equivariant real-analytic gradient map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$.

3.1. Minimal parabolic subgroups. — For more details and complete proofs of the material presented here we refer the reader to Chapter VII in [**Kna02**].

Since $G = K\exp(\mathfrak{p})$ is invariant under the Cartan involution θ of $U^{\mathbb{C}}$, the same holds for its Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Consequently $\mathfrak{g}$ is reductive, i. e. $\mathfrak{g}$ is the direct sum of its center and of the semi-simple subalgebra $[\mathfrak{g}, \mathfrak{g}]$.

Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Lambda} \mathfrak{g}_\lambda$ be the associated restricted root space decomposition. The centralizer $\mathfrak{g}_0$ of $\mathfrak{a}$ in $\mathfrak{g}$ is θ–stable with decomposition $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{m} = \mathcal{Z}_{\mathfrak{k}}(\mathfrak{a})$. On the group level we define $M := \mathcal{Z}_K(\mathfrak{a})$.

Let us fix a choice Λ^+ of positive restricted roots. Then we obtain the nilpotent subalgebra $\mathfrak{n} := \bigoplus_{\lambda\in\Lambda^+} \mathfrak{g}_\lambda$. Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then $AN \subset G$ is a simply-connected solvable closed subgroup of G, isomorphic to the semi-direct product $A \ltimes N$. One checks directly that M stabilizes each restricted root space $\mathfrak{g}_\lambda$; together with the compactness of M this implies that $Q_0 := MAN$ is a closed subgroup of G.

Every subgroup of G which is conjugate to $Q_0 = MAN$ is called a *minimal parabolic subgroup*. A subgroup $Q \subset G$ is called *parabolic* if it contains a minimal parabolic subgroup.

Remark. — The notion of parabolic subgroups of G is independent of the choices made during the construction of Q_0.

Example. — For $\xi \in \mathfrak{p}$ the group $Q := \big\{g \in G;\ \lim_{t\to-\infty} \exp(t\xi)g\exp(-t\xi) \text{ exists in } G\big\}$ is a parabolic subgroup of G. It is a minimal parabolic subgroup if and only if ξ is regular, i. e. if and only if $K_\xi = M$.

If the group G is complex-reductive and connected, then minimal parabolic subgroups of G are the same as Borel subgroups. This motivates the following

Definition 3.1. — We call the G–gradient manifold X *spherical* if a minimal parabolic subgroup of G has an open orbit in X.

Note that X is spherical if and only if $Q_0 = MAN$ has an open orbit in X.

Example. — Let G be a real form of $U^{\mathbb{C}}$ and let $X \subset Z$ be a totally real G–stable submanifold with $\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} Z$. If Z is $U^{\mathbb{C}}$–spherical, then X is G–spherical in the above sense. This can be seen as follows. Since $Q_0^{\mathbb{C}}$ is a parabolic subgroup of $U^{\mathbb{C}} = G^{\mathbb{C}}$ and since Z is spherical, $Q_0^{\mathbb{C}}$ has an open orbit in Z. Since X is maximally totally real, X cannot be contained in the complement of the open $Q_0^{\mathbb{C}}$–orbit in Z, hence we find a point $x \in X$ such that $Q_0^{\mathbb{C}} \cdot x$ is open in Z. Moreover, $Q_0 \cdot x$ is open in $(Q_0^{\mathbb{C}} \cdot x) \cap X$, which implies that X is spherical.

Example. — As a special case of the above example we note that weakly symmetric spaces are spherical gradient manifolds. More precisely, let $G^{\mathbb{C}}$ be connected complex-reductive and let $L^{\mathbb{C}}$ be a complex-reductive compatible subgroup of $G^{\mathbb{C}}$. Let G be a connected compatible real form of $G^{\mathbb{C}}$ such that $L := L^{\mathbb{C}} \cap G$ is a compact real form of $L^{\mathbb{C}}$. According to Theorem 3.11 in [**St08**] the homogeneous manifold $X = G/L$ is a G–gradient manifold. By a result of Akhiezer and Vinberg ([**AV99**], compare also Chapter 12.6 in [**Wo07**]) $X = G/L$ is weakly symmetric if and only if the affine variety $G^{\mathbb{C}}/L^{\mathbb{C}}$ is spherical. This implies that if $X = G/L$ is weakly symmetric, then it is a spherical G–gradient manifold. The converse is false as the next example shows.

Example. — Let U be connected. A special case of Example 2.2 is the case that $Z = U^{\mathbb{C}}$ and $\tau = \sigma = \theta$. Then we have $G = X = U$. Note that $\mu_{\mathfrak{p}} \equiv 0$ separates the K–orbits in X since X is K–homogeneous while in general μ does not separate the U–orbits in Z. Note also that $Q_0 = G$ is the only minimal parabolic subgroup of G and that G itself is the only subgroup of G having an open orbit in X. This explains the necessity to consider minimal parabolic subgroups instead of maximal connected solvable subgroups (which are maximal tori in G in this example).

3.2. Coadjoint orbits. — A class of examples of gradient manifolds is given by coadjoint orbits (see [**HS07c**]). Let $\alpha \in \mathfrak{u}^*$ and let $Z = U \cdot \alpha$ be the coadjoint orbit of α. Identifying $\mathfrak{u}^*$ with $i\mathfrak{u}$ as before, α corresponds to an element $\xi \in i\mathfrak{u}$ and Z corresponds to the orbit of ξ of the adjoint action of U on $i\mathfrak{u}$. Let $P := \{g \in U^{\mathbb{C}};\ \lim_{t\to-\infty} \exp(t\xi) g \exp(-t\xi) \text{ exists in } U^{\mathbb{C}}\}$ denote the parabolic subgroup of $U^{\mathbb{C}}$ associated to ξ. Then the map $Z \to U^{\mathbb{C}}/P$, $u \cdot \xi \mapsto uP$, is a real analytic isomorphism. In particular it defines a complex structure and a holomorphic $U^{\mathbb{C}}$–action on Z. The reader should be warned that this $U^{\mathbb{C}}$–action is not the adjoint action. The form $\omega\big(\eta_Z(\alpha), \zeta_Z(\alpha)\big) = -\alpha\big([\eta, \zeta]\big)$ defines a U–invariant Kähler form on $Z = U \cdot \alpha$ such that the map $\mu\colon Z \to \mathfrak{u}^*$, $\mu(u \cdot \alpha) = -\operatorname{Ad}(u)\alpha$, is a moment map on Z. Identifying Z with U/U_ξ where U_ξ denotes the centralizer of ξ in U, the gradient map with respect to the action of $U^{\mathbb{C}}$ on Z is given by $\mu_{i\mathfrak{u}}\colon U/U_\xi \to i\mathfrak{u}$, $uU_\xi \mapsto -\operatorname{Ad}(u)\xi$. The $U^{\mathbb{C}}$–action on $U \cdot \xi \cong U^{\mathbb{C}}/P$ induces a G–action on $U \cdot \xi$.

Proposition 3.2 **([HS07c]).** — *If $\xi \in \mathfrak{p}$, then $X := K \cdot \xi = G \cdot \xi$ is a Lagrangian submanifold of $Z \cong U \cdot \xi$.*

The G-isotropy at ξ is given by the parabolic subgroup $Q := P \cap G$ of G, so $G \cdot \xi$ is isomorphic to G/Q and to K/K_ξ if $\xi \in \mathfrak{p}$. Note also that G/Q is a compact G–invariant submanifold of $U^{\mathbb{C}}/P$ and in particular a G-gradient manifold with gradient map $\mu_{\mathfrak{p}}\colon K/K_\xi \to \mathfrak{p}$, $\mu_{\mathfrak{p}}(kK_\xi) = -\operatorname{Ad}(k)\xi$.

Example. — Consider the action of $G = \mathrm{SL}(2, \mathbb{R})$ on projective space $Z = \mathbb{P}_1(\mathbb{C})$ induced by the standard representation of G on $\mathbb{C}^2$. Note that G is a compatible subgroup of $U^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ where $U = \mathrm{SU}(2)$. Moreover, Z can be realized as the coadjoint orbit $U^{\mathbb{C}}/B$ where B is the Borel subgroup $B = \left\{ \begin{pmatrix} z & w \\ 0 & z^{-1} \end{pmatrix}; z \in \mathbb{C}^*, w \in \mathbb{C} \right\}$. Then Z can be viewed as a 2-sphere in the 3-dimensional space $i\mathfrak{u}$. The gradient map $\mu_{\mathfrak{p}}$ is the projection onto the 2-dimensional subspace $\mathfrak{p}$ of $i\mathfrak{u}$. The action of K on $i\mathfrak{u}$ is given by rotation around the axes perpendicular to $\mathfrak{p}$. We observe that $\mu_{\mathfrak{p}}$ almost separates the K–orbits, but that it does not separate all K–orbits. This corresponds to the fact that there exist two open orbits with respect to the action of a minimal parabolic subgroup of G.

If $G = U^{\mathbb{C}}$ is complex reductive and acts algebraically on a connected algebraic variety Z, then the fibers of the moment map μ are connected ([**HH96**]). Also, if Z is spherical, then μ globally separates the U–orbits. The example above shows that one cannot expect $\mu_{\mathfrak{p}}$ to separate the K–orbits globally for actions of real-reductive groups due to the non-connectedness of the $\mu_{\mathfrak{p}}$–fibers. Moreover, in the complex case an open orbit of a Borel subgroup is unique and dense in Z while this is no longer true for real-reductive groups.

4. The generic fibers of the restricted gradient map

By equivariance, the moment map $\mu\colon Z \to \mathfrak{u}^*$ maps each orbit $U\cdot z$ onto the orbit $U\cdot\mu(z) \subset \mathfrak{u}^*$. Moreover, the restriction $\mu|_{U\cdot z}\colon U\cdot z \to U\cdot\mu(z)$ is a smooth fiber bundle with fiber $U_{\mu(z)}/U_z$. Theorem 26.5 in [**GS84**] states that generically these fibers are tori; in [**HW90**] this theorem is applied to characterize coisotropic U–actions.

In this section we generalize these results in our context. Let $x \in X$ and let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$ with $\mu_{\mathfrak{p}}(x) \in \mathfrak{a}$. Our goal is to prove that generically the group $M = \mathcal{Z}_K(\mathfrak{a})$ has an open orbit in the fiber $K_{\mu_{\mathfrak{p}}(x)}/K_x$ of $\mu_{\mathfrak{p}}\colon K\cdot x \to K\cdot\mu_{\mathfrak{p}}(x)$. For this we first have to discuss the notion of generic elements in X.

4.1. Generic elements. — There are several natural definitions of generic elements $x \in X$. We could require that the K–orbit through x has maximal dimension, or that the K–orbit through $\mu_{\mathfrak{p}}(x)$ has maximal dimension in $\mu_{\mathfrak{p}}(X)$, or that the rank of $\mu_{\mathfrak{p}}$ in x is maximal. It will turn out that we need all three properties.

Definition 4.1. — The element $x \in X$ is called generic if

1. the dimension of $K\cdot x$ is maximal,
2. the rank of $\mu_{\mathfrak{p}}$ in x is maximal, and
3. the dimension of $K\cdot\mu_{\mathfrak{p}}(x)$ is maximal in $\mu_{\mathfrak{p}}(X)$.

We write X_{gen} for the set of generic elements in X.

Remark. — In the complex case we have $\operatorname{rk}_z \mu = \dim U\cdot z$; hence, condition (2) in Definition 4.1 is superfluous in this case.

For the following lemma we need the analyticity of $\mu_{\mathfrak{p}}$ and of the K–action on X.

Lemma 4.2. — *The set X_{gen} is K–invariant, open and dense in X.*

Proof. — Since X/G is connected, the same is true for X/K. It is then a well-known consequence of the Slice Theorem that the set of points $x \in X$ such that $K\cdot x$ has maximal dimension is open and dense in X (see Theorem 3.1, Chapter IV in [**Bre72**]). Since $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ is real-analytic, its maximal rank set is also open and dense. Hence, $X' := \{x \in X;\ \dim K\cdot x, \operatorname{rk}_x \mu_{\mathfrak{p}} \text{ maximal}\}$ is open and dense in X.

We prove the lemma by showing that $X' \setminus X_{\text{gen}}$ is analytic in X'. Let $x_0 \in X' \setminus X_{\text{gen}}$. Since $\mu_{\mathfrak{p}}$ has constant rank on X', there are local analytic coordinates (x, U) around x_0 in X and (y, V) around $\mu_{\mathfrak{p}}(x_0)$ in $\mu_{\mathfrak{p}}(X)$ in which $\mu_{\mathfrak{p}}$ takes the form $\mu_{\mathfrak{p}}(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$. Since $\mu_{\mathfrak{p}}$ is K–equivariant, U and V may be chosen K–invariant. Since $A := \{y \in V;\ \dim K\cdot y \text{ is not maximal in } V\}$ is analytic in V, we see that $(X' \setminus X_{\text{gen}}) \cap U = \mu_{\mathfrak{p}}^{-1}(A)$ is analytic in U. Thus $X' \setminus X_{\text{gen}}$ is locally analytic in X and since it is closed, it is analytic. $\square$

4.2. The M–action on $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)$. — In this subsection we discuss the restricted gradient map

$$\mu_{\mathfrak{p}}|_{K\cdot x}\colon K\cdot x\to K\cdot\mu_{\mathfrak{p}}(x).$$

Recall that this map is a smooth fiber bundle with fiber $K_{\mu_{\mathfrak{p}}(x)}/K_x$.

Remark. — Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. Then we have $M\subset K_{\mu_{\mathfrak{p}}(x)}$ for every $x\in X$ with $\mu_{\mathfrak{p}}(x)\in\mathfrak{a}$. Note that every K–orbit in X intersects $\mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$.

We will need the following lemma which extends the corresponding result in [**GS84**].

Lemma 4.3. — *For every $x\in X_{\text{gen}}$ we have $[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)},\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]\subset\mathfrak{p}_x$.*

Proof. — By definition of X_{gen} the set

$$E:=\big\{(x,\xi,\eta)\in X_{\text{gen}}\times\mathfrak{k}\times\mathfrak{p};\ \xi\in\mathfrak{k}_{\mu_{\mathfrak{p}}(x)},\eta\in\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}\big\}$$

is a linear subbundle of the trivial bundle $X_{\text{gen}}\times\mathfrak{k}\times\mathfrak{p}\to X_{\text{gen}}$.

Let $\xi\in\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$ and $\eta\in\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}$, and let x_t be a smooth curve in X_{gen} with $x_0=x$. Since $E\to X_{\text{gen}}$ is locally trivial, we find a smooth curve (x_t,ξ_t,η_t) in E with $\xi_0=\xi$ and $\eta_0=\eta$. Since $[\xi_t,\eta_t]\in\mathfrak{p}_{\mu_{\mathfrak{p}}(x_t)}$ for all t and since the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ is induced by a U–invariant inner product on $\mathfrak{u}$, we conclude

$$\big\langle\mu_{\mathfrak{p}}(x_t),[\xi_t,\eta_t]\big\rangle=-\big\langle\big[\xi_t,\mu_{\mathfrak{p}}(x_t)\big],\eta_t\big\rangle=0$$

for all t. Differentiating and evaluating at $t=0$ yields

$$\begin{aligned}0&=\big\langle(\mu_{\mathfrak{p}})_{*,x}\dot{x}_0,[\xi,\eta]\big\rangle+\big\langle\mu_{\mathfrak{p}}(x),[\dot{\xi}_0,\eta]\big\rangle+\big\langle\mu_{\mathfrak{p}}(x),[\xi,\dot{\eta}_0]\big\rangle\\&=\big\langle(\mu_{\mathfrak{p}})_{*,x}\dot{x}_0,[\xi,\eta]\big\rangle-\big\langle\big[\eta,\mu_{\mathfrak{p}}(x)\big],\dot{\xi}_0\big\rangle-\big\langle\big[\xi,\mu_{\mathfrak{p}}(x)\big],\dot{\eta}_0\big\rangle\\&=\big\langle(\mu_{\mathfrak{p}})_{*,x}\dot{x}_0,[\xi,\eta]\big\rangle=g_x\big([\xi,\eta]_X(x),\dot{x}_0\big).\end{aligned}$$

Since X_{gen} is open, every tangent vector $v\in T_xX$ is of the form $v=\dot{x}_0$ for some curve x_t which implies $[\xi,\eta]_X(x)=0$, i. e. $[\xi,\eta]\in\mathfrak{p}_x$. □

Now we are in the position to prove

Proposition 4.4. — *Suppose $x\in X_{\text{gen}}\cap\mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$. Then the orbit $M\cdot x$ is open in $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)\cap(K\cdot x)$.*

Let $x\in X_{\text{gen}}\cap\mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ be given. In order to prove Proposition 4.4 it suffices to show that the map $\mathfrak{m}\to\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}/\mathfrak{k}_x$ is surjective. For this we need some information about $\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$ and $\mathfrak{k}_x$; the idea is of course to apply Lemma 4.3 which gives

$$\big[[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)},\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}],[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)},\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]\big]\subset[\mathfrak{p}_x,\mathfrak{p}_x]\subset\mathfrak{k}_x.$$

Consequently we must determine $\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$, $\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}$ as well as their Lie brackets.

This is most conveniently done via the restricted root space decomposition $\mathfrak{g}=\mathfrak{g}_0\oplus\bigoplus_{\lambda\in\Lambda}\mathfrak{g}_\lambda$ with respect to the maximal Abelian subspace $\mathfrak{a}\subset\mathfrak{p}$. The centralizer $\mathfrak{g}_0$ of $\mathfrak{a}$ in $\mathfrak{g}$ is stable under the Cartan involution θ and decomposes as $\mathfrak{g}_0=\mathfrak{m}\oplus\mathfrak{a}$ where $\mathfrak{m}=\text{Lie}(M)$. For later use we note the following proposition which is proven in Chapter VI.5 of [**Kna02**].

Proposition 4.5. — *For each $\lambda\in\Lambda$ we write $\mathfrak{a}_\lambda\subset\mathfrak{a}$ for the subspace generated by the elements $\big[\xi_\lambda,\theta(\xi_\lambda)\big]$ where $\xi_\lambda\in\mathfrak{g}_\lambda$. Then* $\dim\mathfrak{a}_\lambda=1$ *and $\lambda\big[\xi_\lambda,\theta(\xi_\lambda)\big]\neq0$ for every $0\neq\xi_\lambda\in\mathfrak{g}_\lambda$.*

In order to prove Proposition 4.4 we will first describe the centralizers of $\mu_{\mathfrak{p}}(x)$ in $\mathfrak{k}$ and in $\mathfrak{p}$. For this we introduce the subset $\Lambda(x):=\big\{\lambda\in\Lambda;\ \lambda\big(\mu_{\mathfrak{p}}(x)\big)=0\big\}\subset\Lambda$. We also write $\Lambda^+(x):=\Lambda(x)\cap\Lambda^+$.

Remark. — If $\lambda \in \Lambda(x)$, then $-\lambda \in \Lambda(x)$. If $\lambda_1, \lambda_2 \in \Lambda(x)$ and $\lambda_1 + \lambda_2 \in \Lambda$, then $\lambda_1 + \lambda_2 \in \Lambda(x)$.

Lemma 4.6. — 1. *The centralizer of* $\mu_{\mathfrak{p}}(x)$ *in* $\mathfrak{g}$ *is given by* $\mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Lambda(x)} \mathfrak{g}_\lambda$.

2. *We have* $\mathfrak{k}_{\mu_{\mathfrak{p}}(x)} = \mathfrak{m} \oplus \left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda + \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\}$.
3. *We have* $\mathfrak{p}_{\mu_{\mathfrak{p}}(x)} = \mathfrak{a} \oplus \left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda - \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\}$.

Proof. — In order to prove the first claim let $\xi = \xi_0 + \sum_{\lambda\in\Lambda} \xi_\lambda \in \mathfrak{g}$ and calculate

$$\big[\mu_{\mathfrak{p}}(x), \xi\big] = \sum_{\lambda\in\Lambda} \lambda\big(\mu_{\mathfrak{p}}(x)\big)\xi_\lambda.$$

Hence, ξ centralizes $\mu_{\mathfrak{p}}(x)$ if and only if $\xi_\lambda = 0$ for all $\lambda \notin \Lambda(x)$.

The other two claims follow from (1) together with the fact that $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$ for all $\lambda \in \Lambda$. □

It remains to show that $\left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda + \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\}$ is contained in $\mathfrak{k}_x$ because then Lemma 4.6 implies that $\mathfrak{m} \to \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}/\mathfrak{k}_x$ is surjective which in turn proves Proposition 4.4.

Lemma 4.7. — *We have* $\left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda + \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\} \subset \mathfrak{k}_x$.

Proof. — We will prove this lemma in three steps.

In the first step we prove

$$\mathfrak{p}^x := \bigoplus_{\lambda\in\Lambda(x)} \mathfrak{a}_\lambda \oplus \left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda - \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\} \subset [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}].$$

Let $\lambda \in \Lambda^+(x)$ and $\xi_\lambda \in \mathfrak{g}_\lambda$. Then we have $\xi_\lambda + \theta(\xi_\lambda) \in \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$, and we may choose an element $\eta \in \mathfrak{a}$ with $\lambda(\eta) \neq 0$. Because of

$$\xi_\lambda - \theta(\xi_\lambda) = -\frac{1}{\lambda(\eta)}\big[\xi_\lambda + \theta(\xi_\lambda), \eta\big] \in [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$$

we obtain $\left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda - \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\} \subset [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$.

Moreover,

$$\big[\xi_\lambda, \theta(\xi_\lambda)\big] = -\frac{1}{2}\big[\xi_\lambda + \theta(\xi_\lambda), \xi_\lambda - \theta(\xi_\lambda)\big] \in [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$$

implies $\mathfrak{a}_\lambda \subset [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$.

The second step consists in showing

$$\left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda + \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\} \subset [\mathfrak{p}^x, \mathfrak{p}^x].$$

To see this, let $\lambda \in \Lambda^+(x)$ and $0 \neq \xi_\lambda \in \mathfrak{g}_\lambda$ be arbitrary. Then we have $\xi_\lambda - \theta(\xi_\lambda) \in \mathfrak{p}^x$ and $\big[\xi_\lambda, \theta(\xi_\lambda)\big] \in \mathfrak{a}_\lambda$. Moreover, Proposition 4.5 implies $\lambda\big[\xi_\lambda, \theta(\xi_\lambda)\big] \neq 0$, which gives

$$\xi_\lambda + \theta(\xi_\lambda) = \frac{1}{\lambda\big[\xi_\lambda, \theta(\xi_\lambda)\big]}\Big[\big[\xi_\lambda, \theta(\xi_\lambda)\big], \xi_\lambda - \theta(\xi_\lambda)\Big] \in [\mathfrak{p}^x, \mathfrak{p}^x].$$

In the last step we combine the results obtained so far with Lemma 4.3 and arrive at

$$\left\{\sum_{\lambda\in\Lambda^+(x)} \big(\xi_\lambda + \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda\right\} \subset [\mathfrak{p}^x, \mathfrak{p}^x] \subset \big[[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}], [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]\big] \subset \mathfrak{k}_x,$$

which was to be shown. □

Hence, the proof of Proposition 4.4 is finished.

4.3. An equivalent condition of the separation property. — Proposition 4.4 allows us to formulate an equivalent condition for $\mu_{\mathfrak{p}}$ to separate locally almost the K–orbits which generalizes the notion of K–spherical symplectic manifolds defined in [**HW90**].

Proposition 4.8. — *The gradient map $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits if and only if $\dim(\mathfrak{p}\cdot x)^{\perp} = \dim M - \dim M_x$ for one (and then every) $x \in X_{\mathrm{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$.*

Proof. — Let us suppose first that $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits. By definition, this means that there is an open K–invariant subset $\Omega \subset X$ such that $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)^0 = K^0_{\mu_{\mathfrak{p}}(x)}\cdot x$ for all $x \in \Omega$.

Since X_{gen} is dense, we find an element $x \in \Omega \cap X_{\mathrm{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$. It follows from maximality of $\mathrm{rk}_x\,\mu_{\mathfrak{p}}$ that $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big) \cap X_{\mathrm{gen}}$ is a closed submanifold of X_{gen}. By Lemma 5.1 in [**HS07b**], we obtain $\dim\ker(\mu_{\mathfrak{p}})_{*,x} = \dim(\mathfrak{p}\cdot x)^{\perp}$. Hence, we conclude $\dim K_{\mu_{\mathfrak{p}}(x)}/K_x = \dim(\mathfrak{p}\cdot x)^{\perp}$. Since by Proposition 4.4 the orbit $M\cdot x$ is open in $K_{\mu_{\mathfrak{p}}(x)}\cdot x$, we finally obtain $\dim(\mathfrak{p}\cdot x)^{\perp} = \dim M/M_x = \dim M - \dim M_x$ which was to be shown.

In order to prove the converse let $x \in X_{\mathrm{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ be given. Our assumption implies that $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)$ is a closed submanifold of X of dimension $\dim(\mathfrak{p}\cdot x)^{\perp} = \dim M - \dim M_x$. We conclude that $M\cdot x$ and hence $K_{\mu_{\mathfrak{p}}(x)}\cdot x$ are open in $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)$. Therefore we have $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x)\big)^0 = K^0_{\mu_{\mathfrak{p}}(x)}\cdot x$, which means that $\mu_{\mathfrak{p}}$ separates the K–orbits in X_{gen}. □

Let us note explicitly the following corollary of the proof of Proposition 4.8.

Corollary 4.9. — *If $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits in X, then it almost separates the K–orbits in the dense open set X_{gen}.*

Consequently, if $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits in X, then $\mu_{\mathfrak{p}}$ induces a map $X_{\mathrm{gen}}/K \to \mathfrak{p}/K \cong \mathfrak{a}/W$ whose fibers are discrete.

5. Proof of the main theorem

In the first subsection we review the shifting technique for gradient maps which translates the problem of finding an open Q_0–orbit in X into the problem of finding an open G–orbit in the bigger gradient manifold $X \times (K/M)$. Since G is real-reductive, we may apply the techniques developed in [**HS07b**] to solve the second problem.

Afterwards, it remains to find an open G–orbit in $X \times (K/M)$ under the assumption that $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits. This is done in two steps: First we construct a special gradient map $\widetilde{\mu}_{\mathfrak{p}}$ on $X \times (K/M)$ for which the set of global minima of $\|\widetilde{\mu}_{\mathfrak{p}}\|^2$ can be controlled. This will then be essentially used in the proof of existence of an open Q_0–orbit.

In the final subsection we prove the remaining implication (3) $\Longrightarrow$ (2) in our main theorem: If the minimal parabolic subgroup Q_0 has an open orbit in X, then $\mu_{\mathfrak{p}}$ almost separates the K–orbits.

5.1. The shifting technique. — Since the minimal parabolic subgroup $Q_0 = MAN$ is not compatible, we cannot apply the theory developed in [**HS07b**] in order to link the action of Q_0 on X with the theory of gradient maps. Therefore, we reformulate the problem of finding an open Q_0–orbit in X as the problem of finding an open G–orbit in a larger manifold.

Lemma 5.1. — *Let Q be a parabolic subgroup of G. Then Q has an open orbit in X if and only if G has an open orbit in $X \times (G/Q)$ with respect to the diagonal action.*

Proof. — Recall that the twisted product $G \times_Q X$ is by definition the quotient space of $G \times X$ by the Q–action $q \cdot (g, x) := (gq^{-1}, q \cdot x)$. We denote the element $Q \cdot (g, x) \in G \times_Q X$ by $[g, x]$. Then G acts on $G \times_Q X$ by $g \cdot [h, x] := [gh, x]$, and every G–orbit in $G \times_Q X$ intersects $X \cong \{[e, x];\ x \in X\}$ in a Q–orbit. Thus, the inclusion $X \hookrightarrow G \times_Q X$, $x \mapsto [e, x]$, induces a homeomorphism $X/Q \cong (G \times_Q X)/G$. In particular, Q has an open orbit in X if and only if G has an open orbit in $G \times_Q X$.

The claim follows now from the fact that the map $G \times_Q X \to X \times (G/Q)$, $[g, x] \mapsto (g \cdot x, gQ)$, is a G–equivariant diffeomorphism with respect to the diagonal G–action on $X \times (G/Q)$. To see this, it is sufficient to note that its inverse map is given by $(x, gQ) \mapsto [g, g^{-1} \cdot x]$. □

The gradient map $\mu_{\mathfrak{p}}$ on X induces in a natural way a gradient map on the product $\widetilde{X} := X \times (G/Q)$ as follows. First recall from Section 3.2 that G/Q is a G–invariant closed submanifold of an adjoint U–orbit of an element $\gamma \in \mathfrak{p}$. In particular G/Q is isomorphic to K/K_γ and is equipped with a gradient map $kK_\gamma \mapsto -\operatorname{Ad}(k)\xi$. The gradient maps on X and on K/K_γ induce a gradient map $\widetilde{\mu}_{\mathfrak{p}}$ on $\widetilde{X}$, which is given by the sum of those two gradient maps. Explicitly, we have

$$\widetilde{\mu}_{\mathfrak{p}}(x, kK_\gamma) = \mu_{\mathfrak{p}}(x) - \operatorname{Ad}(k)\gamma.$$

Note that the choice of $\gamma \in \mathfrak{p}$ depends only on the isotropy K_γ. In particular, if Q is a minimal parabolic subgroup of G, or equivalently if K_γ equals the centralizer M of $\mathfrak{a}$ in K, then for every regular $\gamma \in \mathfrak{p}$, the assignment $(x, kM) \mapsto \mu_{\mathfrak{p}}(x) - \operatorname{Ad}(k)\gamma$ defines a gradient map on $\widetilde{X}$.

5.2. The shifted gradient map. — Our goal is to construct a gradient map on $\widetilde{X} = X \times (K/M)$ which enables us to control the minima of the associated function $\|\widetilde{\mu}_{\mathfrak{p}}\|^2$.

Let $\mathfrak{a}_+$ denote the closed Weyl chamber in $\mathfrak{a}$ associated to our choice of positive restricted roots. We generalize an inequality in [**HS07a**] which is a consequence of Kostant's Convexity Theorem ([**Kos73**]).

Lemma 5.2. — *Let $\gamma, \xi \in \mathfrak{a}_+$ and assume that ξ is regular. Then*

$$\|\operatorname{Ad}(k)\gamma - \xi\| \geq \|\gamma - \xi\|$$

for all $k \in K$. The inequality is strict for all $k \notin K_\gamma$.

Proof. — The K-invariance of the inner product implies

$$\|\operatorname{Ad}(k)\gamma - \xi\|^2 - \|\gamma - \xi\|^2 = -2 \cdot \langle \operatorname{Ad}(k)\gamma - \gamma, \xi \rangle.$$

Let $\pi_{\mathfrak{a}}$ denote the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$. Then $\langle \operatorname{Ad}(k)\gamma, \xi \rangle = \langle \pi_{\mathfrak{a}}(\operatorname{Ad}(k)\gamma), \xi \rangle$ and $\pi_{\mathfrak{a}}(\operatorname{Ad}(k)\gamma)$ is contained in the convex hull of the orbit of the Weyl group $W := \mathcal{N}_K(\mathfrak{a})/\mathcal{Z}_K(\mathfrak{a})$ through ξ ([**Kos73**]). Since K acts by unitary operators, we have $\pi_{\mathfrak{a}}(\operatorname{Ad}(k)\gamma) = \gamma$ if and only if $k \in K_\gamma$. Therefore it suffices to show that $\langle \operatorname{Ad}(w)\gamma - \gamma, \xi \rangle < 0$ for all $w \in W$, $w \notin W_\gamma$.

Let λ be a simple restricted root and σ_λ the corresponding reflection. Then either $\sigma_\lambda(\gamma) = \gamma$ or $\sigma_\lambda(\gamma) - \gamma = c \cdot \lambda$ for some $c < 0$. Here we have identified $\lambda \in \mathfrak{a}^*$ with its dual in $\mathfrak{a}$. Since ξ is regular, this implies $\langle \sigma_\lambda(\gamma) - \gamma, \xi \rangle < 0$ if $\sigma_\lambda \notin W_\gamma$.

An arbitrary element $w \in W$ is of the form $w = \sigma_{\lambda_1} \circ \cdots \circ \sigma_{\lambda_k}$ for simple restricted roots $\lambda_1, \ldots, \lambda_k$. Then

$$\begin{aligned}\mathrm{Ad}(w)\gamma - \gamma = &\big(\sigma_{\lambda_1} \circ \cdots \circ \sigma_{\lambda_k}(\gamma) - \sigma_{\lambda_2} \circ \cdots \circ \sigma_{\lambda_k}(\gamma)\big)\\ &+ \big(\sigma_{\lambda_2} \circ \cdots \circ \sigma_{\lambda_k}(\gamma) - \sigma_{\lambda_3} \circ \cdots \circ \sigma_{\lambda_k}(\gamma)\big)\\ &+ \cdots + \big(\sigma_{\lambda_k}(\gamma) - \gamma\big)\end{aligned}$$

is a linear combination of simple restricted roots with negative coefficients and it equals 0 if and only if $\sigma_{\lambda_j} \in \mathcal{W}_\gamma$ for all j. Again, since ξ is regular, this implies $\big\langle \mathrm{Ad}(w)\gamma - \gamma, \xi\big\rangle < 0$ for all $w \in W$, $w \notin W_\gamma$. □

Since each K–orbit in $\mathfrak{p}$ intersects $\mathfrak{a}$ in an orbit of the Weyl group, each K–orbit $K \cdot x$ in X contains an x_0 with $\mu_\mathfrak{p}(x_0) \in \mathfrak{a}_+$. Recall that each $\xi \in \mathfrak{a}_+$ defines a gradient map $\widetilde{\mu}_\mathfrak{p} \colon \widetilde{X} \to \mathfrak{p}$, $\widetilde{\mu}_\mathfrak{p}(x, kM) = \mu_\mathfrak{p}(x) - \mathrm{Ad}(k)\xi$.

Proposition 5.3. — *Let $x_0 \in X_{\mathrm{gen}}$ with $\mu_\mathfrak{p}(x_0) \in \mathfrak{a}_+$. Then there exists a regular $\xi \in \mathfrak{a}_+$, such that*

1. *(x_0, eM) is a global minimum of the function $\|\widetilde{\mu}_\mathfrak{p}\|^2$.*
2. *If $(x, kM) \in \widetilde{X}$ is another global minimum of $\|\widetilde{\mu}_\mathfrak{p}\|^2$, then $\mu_\mathfrak{p}(x) = \mathrm{Ad}(k)\mu_\mathfrak{p}(x_0)$.*

Proof. — If $\mu_\mathfrak{p}(x_0)$ is regular, define $\xi := \mu_\mathfrak{p}(x_0)$. Then $\|\widetilde{\mu}_\mathfrak{p}(x_0, eM)\|^2 = 0$ and (x_0, eM) is a global minimum of $\|\widetilde{\mu}_\mathfrak{p}\|^2$. If (x, kM) is another global minimum, we have $\mu_\mathfrak{p}(x) - \mathrm{Ad}(k)\xi = 0$ and the second claim follows.

Now assume that $\gamma := \mu_\mathfrak{p}(x_0)$ is singular. Let $\lambda_1, \ldots, \lambda_k$ be those simple restricted roots vanishing at γ. Let $\mathfrak{b} := \big\{\eta \in \mathfrak{a};\ \lambda_1(\zeta) = \ldots = \lambda_k(\eta) = 0\big\}$ be the subspace of $\mathfrak{a}$ where these roots vanish. Let $\mathfrak{b}^\perp$ be the orthogonal complement of $\mathfrak{b}$ in $\mathfrak{a}$. Since x_0 is regular, the orbit $K \cdot \gamma$ has maximal dimension in $\mu_\mathfrak{p}(X)$. Therefore $\mu_\mathfrak{p}(X) \cap \mathfrak{a}$ is contained in the union of the finitely many subspaces of $\mathfrak{a}$ where at least k simples restricted roots vanish. Choosing a regular element $\xi \in \gamma + \mathfrak{b}^\perp$ which is sufficiently near γ, we can assure that γ is the unique point in $\mu_\mathfrak{p}(X) \cap \mathfrak{a}_+$ with minimal distance to ξ.

Let $(x, kM) \in \widetilde{X}$ and let $l \in K$ with $\gamma' := \mathrm{Ad}(l)\mu_\mathfrak{p}(k^{-1} \cdot x) \in \mathfrak{a}_+$. With Lemma 5.2 and the definition of ξ we obtain

$$\begin{aligned}\|\widetilde{\mu}_\mathfrak{p}(x, kM)\|^2 = \|\mu_\mathfrak{p}(x) - \mathrm{Ad}(k)\xi\|^2 &= \|\mu_\mathfrak{p}(k^{-1} \cdot x) - \xi\|^2\\ &\geq \|\gamma' - \xi\|^2 \geq \|\gamma - \xi\|^2 = \|\widetilde{\mu}_\mathfrak{p}(x_0, eM)\|^2,\end{aligned}$$

so in particular (x_0, eM) is a global minimum of $\|\widetilde{\mu}_\mathfrak{p}\|^2$. Equality holds if and only if $\gamma' = \gamma$ and $l \in K_{\gamma'} = K_\gamma$. The latter condition gives $\mathrm{Ad}(k)\gamma = \mu_\mathfrak{p}(x)$. □

In Lemma 5.1, we reformulated the property that a parabolic subgroup Q has an open orbit in X as a property on the G-action on the product $X \times (G/Q)$. Now, we translate the condition, that $\mu_\mathfrak{p}$ locally almost separates the K-orbits to a suitable condition on the shifted gradient map $\widetilde{\mu}_\mathfrak{p}$ on the product $X \times (G/Q)$.

Lemma 5.4. — *Let $\xi \in \mathfrak{a}$ and let $\widetilde{\mu}_\mathfrak{p} \colon \widetilde{X} \to \mathfrak{p}$ be the associated gradient map. Let $x_0 \in X$ with $\mu_\mathfrak{p}(x_0) \in \mathfrak{a}_+$ and set $\beta := \mu_\mathfrak{p}(x_0) - \xi = \widetilde{\mu}_\mathfrak{p}(x_0, eM)$. Then the inclusion $\mu_\mathfrak{p}^{-1}\big(\mu_\mathfrak{p}(x_0)\big) \hookrightarrow \widetilde{\mu}_\mathfrak{p}^{-1}(\beta)$, $x \mapsto (x, eM)$, induces an injective continuous map $\Phi \colon \mu_\mathfrak{p}^{-1}\big(\mu_\mathfrak{p}(x_0)\big)/M \to \widetilde{\mu}_\mathfrak{p}^{-1}(\beta)/K_\beta$. If ξ is chosen such that the conclusions of Proposition 5.3 are satisfied, then Φ is a homeomorphism.*

Proof. — First note that the map $\Phi\colon \mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big)/M \to \widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta)/K_\beta$ is well-defined since M is contained in K_β and $K_{\mu_{\mathfrak{p}}(x_0)}$ and since $\mu_{\mathfrak{p}}$ and $\widetilde{\mu}_{\mathfrak{p}}$ are K–equivariant.

For injectivity, let $x, y \in \mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big)$ with $K_\beta\cdot(x, eM) = K_\beta\cdot(y, eM)$. The latter condition implies $M\cdot x = M\cdot y$ since $K_\beta \cap M = M$. This shows injectivity.

Assume that $x_0 \in \mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big)$ satisfies the conclusions of Proposition 5.3 and let $(x, kM) \in \widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$. Then (x, kM) is a global minimum of $\|\widetilde{\mu}_{\mathfrak{p}}\|^2$ which implies $\mu_{\mathfrak{p}}(x) = \mathrm{Ad}(k)\mu_{\mathfrak{p}}(x_0)$. We conclude $\beta = \widetilde{\mu}_{\mathfrak{p}}(x, kM) = \mu_{\mathfrak{p}}(x) - \mathrm{Ad}(k)\xi = \mathrm{Ad}(k)(\mu_{\mathfrak{p}}(x_0) - \xi) = \mathrm{Ad}(k)\beta$. This proves $k \in K_\beta$. Consequently $K_\beta\cdot(x, kM)$ intersects $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big) \times \{eM\}$ and surjectivity follows. Finally, the inclusion $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big) \hookrightarrow \widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$ is continuous and proper, so Φ is continuous and proper which implies that it is a homeomorphism. $\square$

5.3. Existence of an open Q_0–orbit. — Finally we are in the position to prove that Q_0 has an open orbit in X given that $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits.

Let us fix a point $x_0 \in X_{\mathrm{gen}}$ such that $\mu_{\mathfrak{p}}(x_0)$ lies in the closed Weyl chamber $\mathfrak{a}_+$. By virtue of Proposition 5.3 we find a regular element $\xi \in \mathfrak{a}_+$ such that $\widetilde{\mu}_{\mathfrak{p}}\colon X\times(K/M) \to \mathfrak{p}$, $(x, kM) \mapsto \mu_{\mathfrak{p}}(x) - \mathrm{Ad}(k)\xi$, is a G–gradient map and such that $\widetilde{x}_0 := (x_0, eM)$ is a global minimum of $\|\widetilde{\mu}_{\mathfrak{p}}\|^2$. Let $Q_0 = MAN$ be the minimal parabolic subgroup of G associated to ξ. Then we may identify K/M with G/Q_0 as gradient manifolds. Let $\beta := \mu_{\mathfrak{p}}(x_0) - \xi$. By Lemma 5.4 the quotients $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big)/M$ and $\widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta)/K_\beta$ are homeomorphic. This implies that $K_\beta\cdot\widetilde{x}_0$ is open in $\widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$.

As we have already seen in the proof of Lemma 2.1, it suffices to prove $(\mathfrak{p}\cdot\widetilde{x}_0)^\perp \subset \mathfrak{k}\cdot\widetilde{x}_0$, for then the orbit $G\cdot\widetilde{x}_0$ is open in $X\times(G/Q_0)$ which in turn implies that $Q_0\cdot x_0$ is open in X. For this we will show that $\widetilde{\mu}_{\mathfrak{p}}$ has maximal rank in $\widetilde{x}_0$ as follows. The image of $T_{x_0}X \oplus T_{eM}K/M$ under $(\widetilde{\mu}_{\mathfrak{p}})_{*,\widetilde{x}_0}$ coincides with $(\mu_{\mathfrak{p}})_{*,x_0}(T_{x_0}X) + [\mathfrak{k}, \xi]$. Since ξ is regular, we obtain

$$[\mathfrak{k}, \xi] = \left\{ \sum_{\lambda\in\Lambda^+} \big(\xi_\lambda - \theta(\xi_\lambda)\big);\ \xi_\lambda \in \mathfrak{g}_\lambda \right\} = \mathfrak{a}^\perp.$$

We use the decomposition $T_xX = (\mathfrak{k}\cdot x) \oplus (\mathfrak{k}\cdot x)^\perp$ and note that $(\mu_{\mathfrak{p}})_{*,x}$ maps $\mathfrak{k}\cdot x$ into $\mathfrak{a}^\perp$ for all x in a neighborhood of x_0. Since moreover $\mu_{\mathfrak{p}}$ locally almost separates the K–orbits, one would expect that $(\mu_{\mathfrak{p}})_{*,x_0}$ maps a subspace of $T_{x_0}X$ which is transversal to $\mathfrak{k}\cdot x_0$ onto a subspace of $\mathfrak{p}$ which is transversal to $\mathfrak{a}^\perp$. This is the content of the following

Lemma 5.5. — *Assume that $\mu_{\mathfrak{p}}$ locally almost separates the K-orbits. For every $x \in X_{\mathrm{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ we have $(\mu_{\mathfrak{p}})_{*,x}((\mathfrak{k}\cdot x)^\perp) \cap \mathfrak{a}^\perp = \{0\}$.*

Proof. — Since x is generic, there exists an open neighborhood $V \subset X$ of x such that the rank of $\mu_{\mathfrak{p}}$ is constant on V. We conclude that $V \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ is a submanifold of V and that the image $\mu_{\mathfrak{p}}\big(V \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})\big)$ is an open subset of the linear subspace $\mathfrak{b} := \bigcap_{\lambda\in\Lambda(x)} \ker(\lambda) \subset \mathfrak{a}$. Moreover, we have $\mu_{\mathfrak{p}}(V)$ is an open subset of $K\cdot\mathfrak{b} \cong K \times_{K_{\mu_{\mathfrak{p}}(x)}} \mathfrak{b} = (K/K_{\mu_{\mathfrak{p}}(x)}) \times \mathfrak{b}$.

Since $\mu_{\mathfrak{p}}$ separates the K–orbits and since x is generic, we have $\ker(\mu_{\mathfrak{p}})_{*,x} = (\mathfrak{p}\cdot x)^\perp \subset \mathfrak{k}\cdot x$ which implies that $(\mu_{\mathfrak{p}})_{*,x}$ is injective on $(\mathfrak{k}\cdot x)^\perp$. Consequently, $\mu_{\mathfrak{p}}$ induces an injective immersion $V/K \to \mathfrak{b}$, therefore $(\mu_{\mathfrak{p}})_{*,x}$ maps $(\mathfrak{k}\cdot x)^\perp$ bijectively onto $\mathfrak{b}$. Since $\mathfrak{b}\cap\mathfrak{a}^\perp = \{0\}$, the claim follows. $\square$

We conclude from Lemma 5.5 that the image of $(\widetilde{\mu}_{\mathfrak{p}})_{*,\widetilde{x}_0}$ is given by $(\mu_{\mathfrak{p}})_{*,x_0}\big((\mathfrak{k}\cdot x_0)^\perp\big) \oplus \mathfrak{a}^\perp$. Since x_0 is generic, the dimension of $(\mu_{\mathfrak{p}})_{*,x}\big((\mathfrak{k}\cdot x)^\perp\big)$ is the same for all x in a neighborhood of x_0. Furthermore, every K–orbit in $X\times(K/M)$ intersects $X\times\{eM\}$, thus the rank of $\widetilde{\mu}_{\mathfrak{p}}$

is constant in a neighborhood of $\widetilde{x}_0$. Consequently, the rank of $\widetilde{\mu}_{\mathfrak{p}}$ must be maximal in $\widetilde{x}_0$. Together with the fact that $K_\beta \cdot \widetilde{x}_0$ is open in $\widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$ this yields

$$(\mathfrak{p} \cdot \widetilde{x}_0)^\perp = T_{\widetilde{x}_0}\widetilde{\mu}_{\mathfrak{p}}^{-1}(\beta) = \mathfrak{k}_\beta \cdot \widetilde{x}_0 \subset \mathfrak{k} \cdot \widetilde{x}_0.$$

Therefore we obtain $T_{\widetilde{x}_0}\widetilde{X} = \mathfrak{p} \cdot \widetilde{x}_0 \oplus (\mathfrak{p} \cdot \widetilde{x}_0)^\perp \subset \mathfrak{p} \cdot \widetilde{x}_0 + \mathfrak{k} \cdot \widetilde{x}_0$ which shows that $G \cdot \widetilde{x}_0$ is open in $\widetilde{X}$.

This proves the implication (1) $\Longrightarrow$ (3) of our our main theorem and gives in addition a precise description of the set of open Q_0-orbits in X.

Theorem 5.6. — *Suppose that $\mu_{\mathfrak{p}}$ locally almost separates the K-orbits. Let $x_0 \in X_{\mathrm{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a}_+)$ be given, let ξ be the element from Proposition 5.3, and let Q_0 be the minimal parabolic subgroup of G associated to ξ. Then $Q_0 \cdot x_0$ is open in X.*

The same method of proof gives the following

Proposition 5.7. — *Suppose that $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ locally almost separates the K–orbits. Let $x \in X_{\mathrm{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ and let Q be the parabolic subgroup of G associated to $\beta := \mu_{\mathfrak{p}}(x)$. Then $Q \cdot x$ is open in X.*

Proof. — In order to show that $Q \cdot x$ is open in X, it suffices to show that $G \cdot (x, eQ)$ is open in $X \times (G/Q)$. For this we note that $G/Q \cong K/K_\beta$ as a K–manifold and that for the shifted gradient map $\widetilde{\mu}_{\mathfrak{p}}\colon X \times (K/K_\beta \to \mathfrak{p}$, $(x, kK_\beta) \mapsto \mu_{\mathfrak{p}}(x) - \mathrm{Ad}(k)\beta$ the element (x, eK_β) lies in $\widetilde{\mathcal{M}}_{\mathfrak{p}}$. Then the same arguments as above apply to show that $G \cdot (x, eK_\beta)$ is open. □

5.4. Proof of (3) $\Longrightarrow$ (2)**.** — In this subsection we complete the proof of our main theorem by showing the remaining non-trivial implication.

Proposition 5.8. — *Suppose that Q_0 has an open orbit in X. Then $\mu_{\mathfrak{p}}$ almost separates the K–orbits.*

Proof. — Let $x_0 \in X$ be given. We must show that $K_{\mu_{\mathfrak{p}}(x_0)} \cdot x_0$ is open in $\mu_{\mathfrak{p}}^{-1}\big(\mu_{\mathfrak{p}}(x_0)\big)$. Let $\gamma := \mu_{\mathfrak{p}}(x_0)$ and let Q be the parabolic subgroup of G associated to γ. Recall that $G/Q \cong K/K_\gamma$ is a G-gradient space with gradient map $kK_\gamma \mapsto -\mathrm{Ad}(k)\gamma$. Consider the shifted gradient map $\widetilde{\mu_{\mathfrak{p}}}\colon X \times (K \cdot \gamma) \to \mathfrak{p}$, $(x, kK_\gamma) \mapsto x - \mathrm{Ad}(k)\gamma$. Since the minimal parabolic subgroup Q_0 has an open orbit in X, the same is true for Q. Hence G has an open orbit in $X \times (K/K_\gamma)$ by Lemma 5.1.

By definition of γ, we have $\widetilde{\mu}_{\mathfrak{p}}(x_0, \gamma) = 0$. Consider the set of semistable points $\mathcal{S}_G(\widetilde{\mu}_{\mathfrak{p}}^{-1}(0)) = \{\widetilde{x} \in \widetilde{X};\ \overline{G \cdot \widetilde{x}} \cap \widetilde{\mu}_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$. It is open in $\widetilde{X}$ ([**HS07c**]) and contains (x_0, γ).

By analyticity of the action, the union V of the open G-orbits in $\mathcal{S}_G(\widetilde{\mu}_{\mathfrak{p}}^{-1}(0))$ is dense in $\mathcal{S}_G(\widetilde{\mu}_{\mathfrak{p}}^{-1}(0))$. We note also that the union of the open G–orbits is locally finite in $\mathcal{S}_G(\widetilde{\mu}_{\mathfrak{p}}^{-1}(0))$ which can be seen as follows. For every $p \in \widetilde{\mu}_{\mathfrak{p}}^{-1}(0)$ there exists a slice neighborhood $G \cdot S \cong G \times_{G_x} S$ where G_x is a compatible subgroup of G and S can be viewed as an open neighborhood of 0 in a G_x–representation space. Since G_x has at most finitely many open orbits in this representation space, we conclude that only finitely many open G–orbits intersect the open set $G \cdot S$ which shows that the union of the open G–orbits in $\mathcal{S}_G(\widetilde{\mu}_{\mathfrak{p}}^{-1}(0))$ is locally finite.

Let W be the union of open G-orbits which contain (x_0, γ) in their closure and let $\overline{W}$ be the closure of W in $\mathcal{S}_G(\widetilde{\mu}_{\mathfrak{p}}^{-1}(0))$. Then W consists of only finitely many open G–orbits and consequently $\overline{W}$ contains an open neighborhood of (x_0, γ). By Corollary 11.18 in [**HS07b**], $\overline{W}$ intersects $\widetilde{\mu}_{\mathfrak{p}}^{-1}(0)$ in $K \cdot (x_0, \gamma)$. Therefore $K \cdot (x_0, \gamma)$ is isolated in $\widetilde{\mu}_{\mathfrak{p}}^{-1}(0)$ which shows that the quotient $\widetilde{\mu_{\mathfrak{p}}}^{-1}(0)/K$ is discrete. Then $\mu_{\mathfrak{p}}^{-1}(\gamma)/M$ is discrete by Lemma 5.4 which means that the M–orbits in $\mu_{\mathfrak{p}}^{-1}(\gamma)$ are open. But $M < K^\gamma$ so the K^γ–orbits are open in $\mu_{\mathfrak{p}}^{-1}(\gamma)$ as well. □

This completes the proof of Theorem 1.

Corollary 5.9. — *Let X be a spherical G–gradient manifold. Then every G–stable real-analytic submanifold Y of X is also spherical.*

Proof. — The claim follows from the facts that Y is a G–gradient manifold with respect to $\mu_{\mathfrak{p}}|_Y$ and that $\mu_{\mathfrak{p}}|_Y$ almost separates the K–orbits in Y since this is true for $\mu_{\mathfrak{p}}$. □

Corollary 5.10. — *If one G–gradient map locally almost separates the K–orbits in X, then every G–gradient map on X almost separates the K–orbits.*

6. Applications

6.1. Homogeneous semi-stable spherical gradient manifolds. — Let $G = K\exp(\mathfrak{p})$ be connected real-reductive and let X be a spherical G–gradient manifold with gradient map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$. We have seen in Lemma 2.1 that G has an open orbit in X. In this subsection we consider the case that $X = G/H$ is homogeneous. In addition, we suppose that X is semi-stable, i. e. that $X = \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$ holds. Consequently, we may assume that H is of the form $H = K_H \exp(\mathfrak{p}_H)$ with $K_H = K \cap H$ and $\mathfrak{p}_H = \mathfrak{p} \cap \mathfrak{h}$.

Remark. — The class of homogeneous semi-stable spherical gradient manifolds generalizes the class of homogeneous affine spherical varieties in the complex setting.

Let $\mathfrak{p} = \mathfrak{p}_H \oplus \mathfrak{p}_H^\perp$ be a K_H–invariant decomposition; then we have the Mostow decomposition $G/H \cong K \times_{K_H} \mathfrak{p}_H^\perp$ (see Theorem 9.3 in [**HS07b**] for a proof which uses gradient maps). Since X is spherical, we conclude from Theorem 1 that the Mostow gradient map $\mu_{\mathfrak{p}}\colon G/H \cong K \times_{K_H} \mathfrak{p}_H^\perp \to \mathfrak{p}$, $[k, \xi] \mapsto \mathrm{Ad}(k)\xi$, almost separates the K–orbits. In other words, the inclusion $\mathfrak{p}_H^\perp \hookrightarrow \mathfrak{p}$ induces a map $\mathfrak{p}_H^\perp/K_H \to \mathfrak{p}/K$ which has discrete fibers. This discussion proves the following

Proposition 6.1. — *Let $X = G/H$ be a semi-stable homogeneous G–gradient manifold and suppose that $H = K_H \exp(\mathfrak{p}_H)$ is compatible in $G = K\exp(\mathfrak{p})$. Then X is spherical if and only if the map $\mathfrak{p}_H^\perp/K_H \to \mathfrak{p}/K$ induced by the inclusion $\mathfrak{p}_H^\perp \hookrightarrow \mathfrak{p}$ has discrete fibers.*

Example. — For $H = \{e\}$ we have $K_H = \{e\}$ and $\mathfrak{p}_H^\perp = \mathfrak{p}$. Consequently, $X = G$ is spherical if and only if the quotient map $\mathfrak{p} \to \mathfrak{p}/K$ has discrete fibers, i. e. if and only if K acts trivially on $\mathfrak{p}$.

Finally, we show that reductive symmetric spaces are spherical. Recall that G/H is a reductive symmetric space if there is an involutive automorphism τ on G such that $(G^\tau)^0 \subset H \subset G^\tau$ holds. In this situation we may assume without loss of generality that τ commutes with the Cartan involution θ. Hence, $H = K^\tau \exp(\mathfrak{p}^\tau)$ is compatible. In order to show that $X = G/H$ is spherical, we must prove that $\mathfrak{p}^{-\tau}/K^\tau \to \mathfrak{p}/K$ has discrete fibers. From $[\mathfrak{p}^{-\tau}, \mathfrak{p}^{-\tau}] \subset \mathfrak{k}^\tau$ we conclude that every K^τ–orbit in $\mathfrak{p}^{-\tau}$ intersects a maximal Abelian subspace $\mathfrak{a}_0 \subset \mathfrak{p}^{-\tau}$ in an orbit of the finite group $W_0 := \mathcal{N}_{K^\tau}(\mathfrak{a}_0)/\mathcal{Z}_{K^\tau}(\mathfrak{a}_0)$. Extending $\mathfrak{a}_0$ to a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ we see that $\mathfrak{p}^{-\tau}/K^\tau \cong \mathfrak{a}_0/W_0 \to \mathfrak{a}/W \cong \mathfrak{p}/K$ has indeed finite fibers. Therefore we have proven the following

Proposition 6.2. — *Let $X = G/H$ be a semi-stable homogeneous gradient manifold. If H is a symmetric subgroup of G, then the Mostow gradient map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ has finite fibers.*

6.2. Relation to multiplicity-free representations. — Let X be a real-analytic G–gradient manifold. Then G acts linearly on the space $\mathcal{C}^\omega(X)$ of complex-valued real-analytic functions on X. Since G is a compatible subgroup of some complex-reductive group $U^{\mathbb{C}}$, we observe that G embeds as a closed subgroup into its complexification $G^{\mathbb{C}}$. Moreover, if G contains no non-compact Abelian factors, then $G^{\mathbb{C}}$ is complex-reductive.

Proposition 6.3. — *Suppose that G acts properly on X and that $G^{\mathbb{C}}$ is complex-reductive. If the G–representation on $\mathcal{C}^\omega(X)$ is multiplicity-free, then X is spherical.*

Proof. — As is proven in [**Hei93**], there exists a Stein $G^{\mathbb{C}}$–manifold $X^{\mathbb{C}}$ such that X admits a G–equivariant embedding as a closed maximally totally real submanifold into $X^{\mathbb{C}}$. According to the example discussed in Section 2.2 it suffices to show that $X^{\mathbb{C}}$ is $G^{\mathbb{C}}$–spherical.

In order to see this, note that the restriction mapping $\mathcal{O}(X^{\mathbb{C}}) \to \mathcal{C}^\omega(X)$ is injective and G–equivariant. This implies that the G– (and hence also the $G^{\mathbb{C}}$–)representation on $\mathcal{O}(X^{\mathbb{C}})$ is multiplicity-free. Therefore, Theorem 2 in [**AH04**] applies to show that $X^{\mathbb{C}}$ is spherical which finishes the proof. □

Remark. — In Proposition 6.3 properness of the G–action on X is needed to guarantee the existence of the complexification $X^{\mathbb{C}}$. If $X = G/H$ is homogeneous, then we may take $X^{\mathbb{C}} := G^{\mathbb{C}}/H^{\mathbb{C}}$ and the same argument as above shows: If the G–representation on $\mathcal{C}^\omega(G/H)$ is multiplicity-free, then G/H is spherical.

Even if we assume that G acts properly on X, the converse of Proposition 6.3 does not hold as the following example shows.

Example. — Let $G = K$ be a compact Lie group acting by left multiplication on $X = K$. Then $\mu_{\mathfrak{p}} \equiv 0$ separates the K–orbits in X but the K–representation on $\mathcal{C}^\omega(K)$ is not multiplicity-free which can be deduced from the Peter-Weyl Theorem.

However, there is a special class of real-reductive Lie groups for which the proof of the complex multiplicity-freeness result generalizes to the real situation. A real-reductive Lie group G belongs to this class if the minimal parabolic subalgebras $\mathfrak{q}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ are solvable, i. e. if $\mathfrak{m}$ is Abelian.

Example. — Among the classical semi-simple Lie groups this is the case e. g. for $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{SU}(p, p)$, $\mathrm{SO}(p, p)$ and $\mathrm{SO}(p, p + 1)$ (see Appendix C.3 in [**Kna02**]).

Lemma 6.4. — *Let X be a spherical G–gradient manifold. If the minimal parabolic subalgebras of $\mathfrak{g}$ are solvable, then the G–representation on $\mathcal{C}^\omega(X)$ is multiplicity-free.*

Proof. — We must show that $\dim \mathrm{Hom}_G\big(V, \mathcal{C}^\omega(X)\big) \leq 1$ holds for every complex finite-dimensional irreducible G–module V. Let $Q_0 = MAN$ be a minimal parabolic subgroup of G and let V be a complex finite-dimensional irreducible G–module. By Engel's Theorem the space V^N of N–invariant vectors has positive dimension. The restriction map induces a linear map

$$\mathrm{Hom}_G\big(V, \mathcal{C}^\omega(X)\big) \to \mathrm{Hom}_{MA}\big(V^N, \mathcal{C}^\omega(X)^N\big),$$

which is injective since V^N generates V as a G–module. Hence, it is enough to show $\dim \mathrm{Hom}_{MA}\big(V^N, \mathcal{C}^\omega(X)^N\big) \leq 1$. Let us assume the contrary. Then there are linearly independent functions $f_1, f_2 \in \mathcal{C}^\omega(X)^N$ which transform under the same character of the Abelian group M^0A. Consequently, the quotient f_1/f_2 is a real-analytic function defined on the dense open set $\{f_2 \neq 0\}$ and invariant under $Q_0^0 = M^0AN$. Since this contradicts the assumption that Q_0 has an open orbit in X, the proof is finished. □

6.3. Open Borel-orbits are Stein. — In this subsection we consider the holomorphic situation, i. e. $G = U^{\mathbb{C}}$ is complex-reductive and acts holomorphically on the Kähler manifold Z such that the U–action is Hamiltonian with moment map $\mu\colon Z \to \mathfrak{u}^*$. In Section 5 we have given a new proof of the following result of Brion.

Theorem 6.5. — *The moment map $\mu\colon Z \to \mathfrak{u}^*$ separates the U–orbits in Z if and only if Z is spherical, i. e. if a Borel subgroup $B \subset G$ has an open orbit in Z.*

In this subsection we will show that our proof further implies that the open B–orbit in Z is Stein.

Proposition 6.6. — *If the moment map $\mu\colon Z \to \mathfrak{u}^*$ separates the U–orbits in Z, then the open B–orbit in Z is Stein.*

Proof. — Let $z \in Z$ be a generic element and let $Q \subset G$ be the parabolic subgroup associated to $\mu(z)$. Consequently, the zero fiber of the shifted moment map on the Kähler manifold $Z \times (G/Q)$ is non-empty. We may assume without loss of generality that the element $(z, eQ) \in Z \times (G/Q)$ is contained in this zero fiber. By Proposition 5.7 the orbit $G \cdot (z, eQ)$ is open in $Z \times (G/Q)$ which in turn implies that $Q \cdot z$ is open in Z. Moreover, since (z, eQ) lies in the zero fiber of a moment map, the isotropy $G_{(z,eQ)} = G_z \cap Q = Q_z$ is complex-reductive which proves that $Q \cdot z \cong Q/Q_z$ is Stein (see Theorem 5 in [**MM60**]). The open B–orbit in Z must be contained in $Q \cdot z$ and is therefore holomorphically separable. Applying a result of Huckleberry and Oeljeklaus ([**HO86**]) we finally see that the open B–orbit is Stein. □

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